

LOGARITHMS OF IMAGINARY NUMBERS IN RECTANGULAR FORM: A NEW TECHNIQUE

Ashwani K Thukral
 Department of Botanical and Environmental Sciences
 Guru Nanak Dev University University, Amritsar -143005, India

ABSTRACT

Logarithms of imaginary numbers are defined by Euler's equation in polar form with real X-axis and imaginary Y-axis. In the present paper, a new concept of logarithms is developed from the basic principles to derive the logarithms of imaginary numbers in rectangular form with both X and Y axes being imaginary. Logarithms of imaginary numbers to a real base with an imaginary multiplicative coefficient are real numbers. This concept generalizes the multiplication and division of real and imaginary numbers using logarithms in rectangular form.

Keywords: Rectangular and polar form logarithms, Euler's equation, imaginary exponential function, logarithmic multiplication and division of imaginary numbers, computational sciences.

INTRODUCTION

The concept of logarithms was introduced by John Napier (1550-1617), a Scottish mathematician (Lexa, 2013). It took Napier about twenty years to complete his tables, which laid the foundation of a new computational technique to comprehend numerals of transcendental size (Clark and Montelle, 2011). Napier provided an arithmetic measure of a geometric ratio (Roegel, 2012). Several prominent scientists like Joost Burgi, Johannes Kepler, Gregoire de Saint-Vincent, Leonard Euler and others contributed to the development of logarithms (Lexa, 2013; Wikipedia, 2014a,b). The term "imaginary" for imaginary numbers was coined by Rene Descartes in 1637 (Wikipedia, 2014c). Gottfried Leibnitz and John Bernoulli-I were the first scientists who initiated discussion on the logarithms of real negative, and imaginary numbers in the year 1745 (Bal, 2014). Euler (1749) defined the exponential function and related it to the natural logarithm. He proved that Leibnitz was correct in claiming that the logarithms of negative numbers are imaginary (Cajori, 1913). Leibnitz and Newton gave the presently used terminology of the logarithms (Lefort, 2013). The logarithmic function has become indispensable to the understanding and development of science and technology, and computational techniques.

Neparian logarithms of real positive numbers (x) are defined as, $\log_b x = y$, such that $b^y = x$,

where y is a real number, and b is a real positive number base of the logarithm (generally 10, or 2 or e). In its integral form, the natural logarithmic function is defined as

$$\ln(x) = \int_1^x (1/t) dt, x > 0 .$$

Natural logarithm is the inverse function of natural exponential function. Logarithms were initially defined for all real positive numbers, until Euler described logarithms of complex and negative numbers, which expanded the field of logarithms.

Complex numbers can be represented using an Argand diagram, in which X-axis is a real axis and Y-axis is an imaginary axis (Fig. 1). A point in the Argand diagram represents a complex number, $x + iy$, where x and y are real numbers, and $i = \sqrt{-1}$.

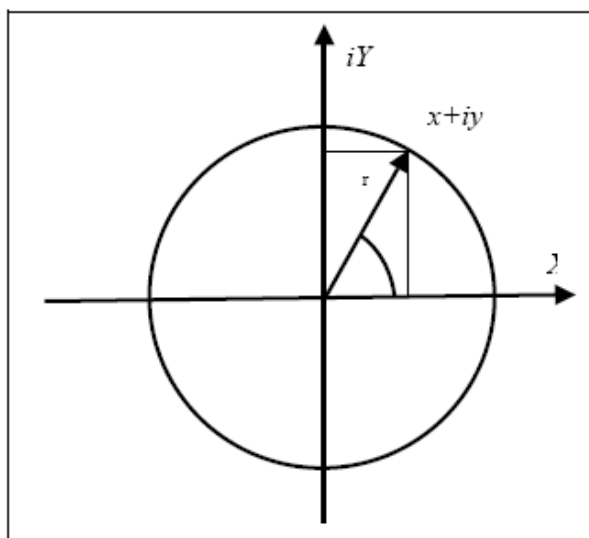


Fig. 1. Argand diagram for logarithms of complex numbers.

*Corresponding author email: akthukral.gndu@gmail.com

Euler (1749) gave a formula known as Euler's identity (Cajori, 1913):

$$e^{i\theta} = \cos x + i \sin x,$$

where, θ is the polar angle in radians between X-axis and the line joining the point (x, iy) with the origin. For $\theta = \pi$,

$$e^{i\pi} + 1 = 0$$

This defined the logarithm of (-1),

$$\log_e(-1) = i\pi.$$

Logarithms of complex numbers may be obtained by converting the rectangular coordinates of the complex number (x, iy) into polar coordinates as follows:

$$x + iy = re^{i\theta},$$

where, r is the magnitude of the radial line for the coordinates (x, iy) , and θ is the polar angle in radians as given by,

$$r = \sqrt{x^2 + y^2},$$

$$\theta = \tan^{-1} \frac{y}{x}.$$

Thus started the field of logarithms of complex numbers. The logarithm of a complex number is obtained as follows:

$$r = y,$$

$$\log_e(x + iy) = \log_e r + i\theta.$$

The logarithms of imaginary numbers are obtained from the above equation with $x = 0$, and

$$\theta = \pi/2 \text{ radians } (\approx 1.570796 \text{ radians}),$$

$$\log_e(iy) = \log_e y + i(\pi/2).$$

This paper is intended to develop a general concept of logarithms applicable to both real and imaginary numbers on a rectangular coordinate system.

1. Logarithms of imaginary numbers

Recently, Thukral and Parkash (2014) generalized the logarithms of real positive numbers and real negative numbers in a rectangular coordinate system as follows:

$$\log_{c,b}(cx) = y, \text{ such that } c(b^y) = cx; x > 0, b > 1, \quad (1.1)$$

where c is +1 for logarithms of real positive numbers, and -1 for logarithms of real negative numbers. In this paper the author now extends this generalized concept to the logarithms of imaginary numbers. Putting the coefficient c equal to (+i) or (-i) in equation (1.1), logarithms of positive imaginary numbers and negative imaginary numbers on rectangular coordinate system may be defined:

Logarithms of positive imaginary numbers

$$\log_{+i,b}(+i)x = y, \quad (1.2)$$

such that $(+i)b^y = (+i)x$, $x > 0, b > 1$, and

logarithms of negative imaginary numbers

$$\log_{-i,b}(-i)x = y, \quad (1.3)$$

such that $(-i)(b^y) = (-i)x$, $x > 0, b > 1$.

There thus follows the equivalence between the logarithms of positive and negative imaginary numbers (equations 1.2, 1.3):

$$\log_{+i,b}(+i)x = \log_{-i,b}(-i)x, x > 0, b > 1.$$

In a rectangular coordinate system with both X and Y axes are real, the area (integral) under the curve above X axis is real positive, and area above the curve under X axis is real negative (Fig. 2). However, since the reciprocals of imaginary numbers will also be imaginary numbers, that is, $1/ix = -ix$ and $1/(-ix) = ix$, the graph between ix and $1/ix$ can be better plotted on imaginary-imaginary (Im-Im) plot. Representing Im-Im axes as iX and iY , the nature of area (integral) under or above the curve gets reversed than in the Real-Real axes (X and Y) plot (Fig. 3).

A graph of the reciprocals of imaginary numbers on Im-Im plot shows the position of the hyperbolas (Fig. 4). Since, $(1/i) = -i$, the curve appears in the IV quadrant for positive imaginary numbers and in the II quadrant for negative imaginary numbers. The integral form of natural imaginary logarithms in rectangular plane will be

$$\log_{c,e}(cx) = \int_c^{cx} (1/t)dt, x > 0, c = +i, \text{ or, } c = -i \quad (1.4)$$

By substitution, the integral for logarithm of positive imaginary numbers will be

$$\log_{+i,e}(+i)x = \int_{(+i)1}^{(+i)x} (1/(it))dit = \int_{(+i)1}^{(+i)x} (-i/t)dit$$

$$= -i^{2 \frac{(+i)x}{(+i)1}} \int (1/t) dt, x > 0 \tag{1.5}$$

$$\log_{+i,e} (+i)x = \int_{(+i)1}^{(+i)x} (1/t) dt, x > 0.$$

It is seen that multiplication of lower and upper limits of the interval of integration (equations 1.4, 1.5) with a constant does not change the logarithm of a number,

$$\log_{c,e} cx = \int_c^{cx} (1/t) dt, x > 0,$$

$$\log_{c,e} cx = \int_1^{cx} (1/t) dt - \int_1^c (1/t) dt = \log_e cx - \log_e c,$$

$$\log_e c + \log_e x - \log_e c = \log_e x.$$

As for example, it may be verified from the computation of area under the curve with the help of a software, that

$$\ln(2) = \int_1^2 (1/t) dt = \int_2^4 (1/t) dt = \int_{0.5}^1 (1/t) dt = \int_{-1}^{-0.5} (1/t) dt = 0.693$$

Therefore, the logarithms of imaginary numbers will be the same as for positive real numbers,

$$\log_{+i,e} (+i)x = \int_1^x (1/t) dt, x > 0. \tag{1.6}$$

Similarly, the integral equation for logarithms of negative imaginary numbers will be

$$\log_{-i,e} (-i)x = - \int_{-x}^{-1} (1/t) dt, x > 0. \tag{1.7}$$

The logarithms of both positive and negative imaginary numbers will be real numbers (equations 1.6, 1.7).

A graph of logarithm of imaginary numbers in the rectangular coordinate system is given in figure 5. If the base of the function is Euler's constant e , then,

$$i(e^y) = ix, \text{ that is, } \log_{i,e} (ix) = y, x > 0, \text{ and}$$

$$-i(e^y) = -ix, \text{ that is, } \log_{-i,e} (-ix) = y, x > 0.$$

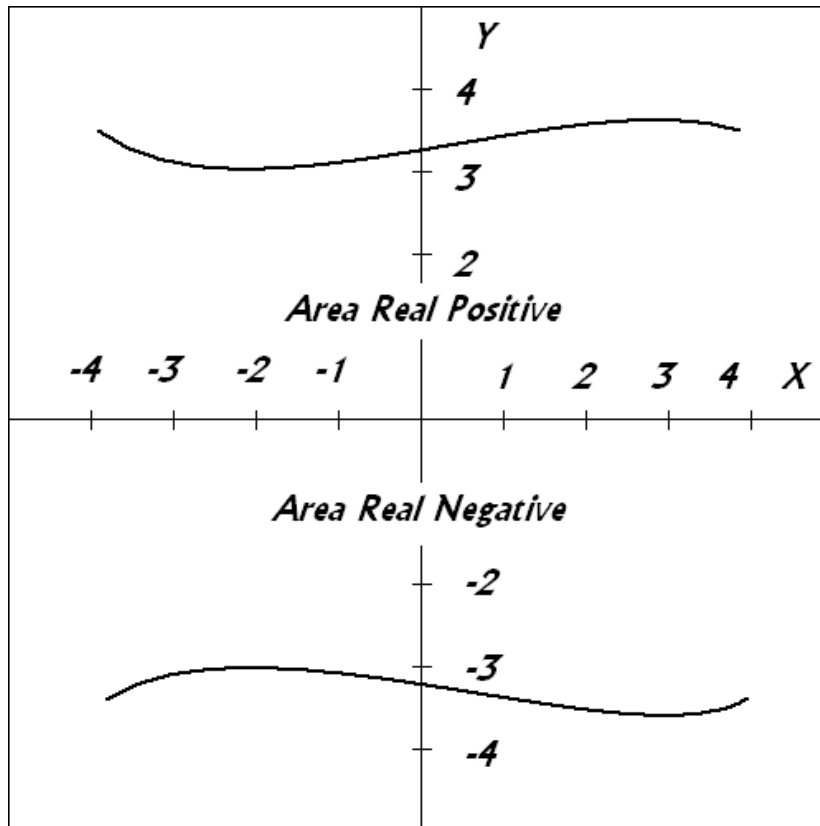


Fig. 2. Area graph for both axes being real (X, Y) (Re-Re plot).

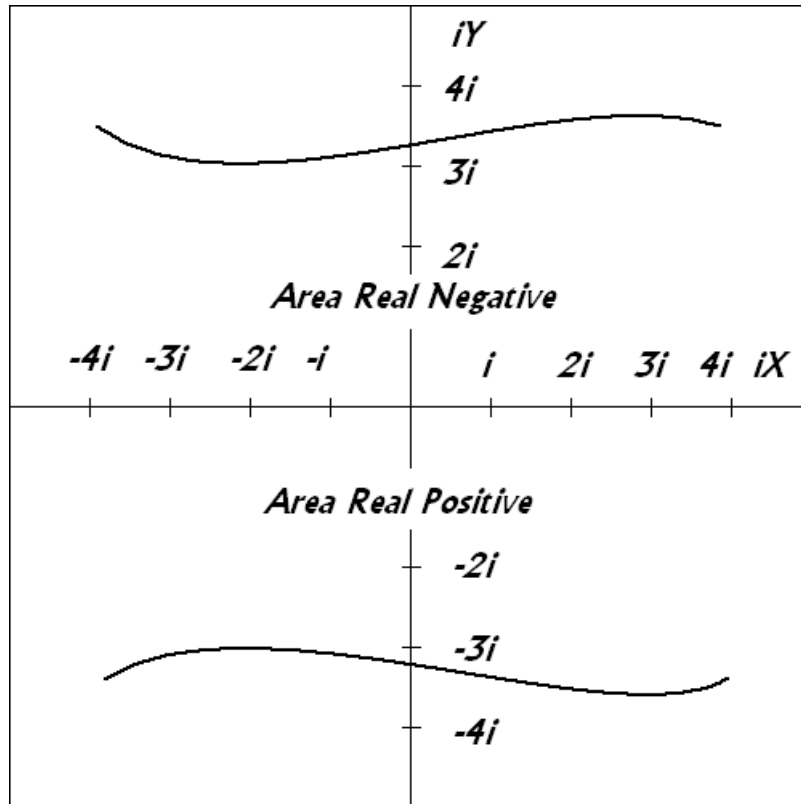


Fig. 3. Area graph for both axes being imaginary (iX, iY) (Im-Im plot).

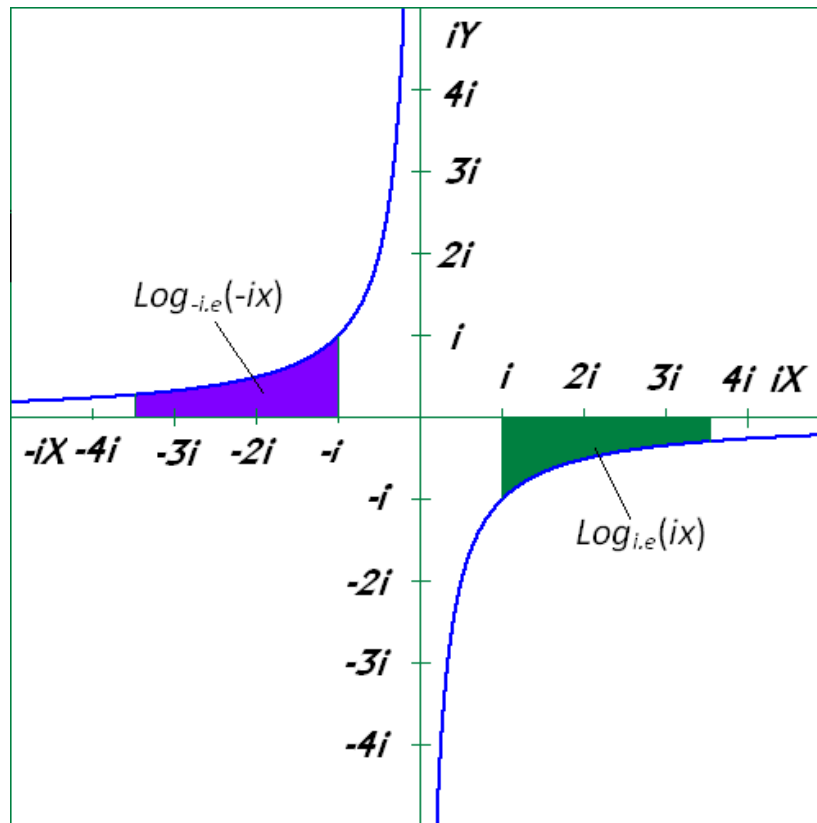


Fig. 4. Curves for hyperbolas transcribed by $y = 1/ix$ in rectangular plane with both X and Y axes being imaginary (Im-Im plot).

Table 1. Logarithms of some numbers in Euler’s polar form and the proposed rectangular form.*

Imaginary number	Euler’s polar form logarithm (Complex number)	Rectangular form logarithm with imaginary number base as per the present concept (Real number)
$(i)x$	$\log_e (i)x$	$\log_{i,e} (i)x$
$0.5 i$	$-0.693+1.570 i$	-0.693
i	$0+1.570 i$	0
$1.5 i$	$0.405+1.570 i$	0.405
$2 i$	$0.693+1.570 i$	0.693
$(-i)x$	$\log_e (i)x$	$\log_{-i,e} (-i)x$
$-0.5 i$	$-0.693-1.570 i$	-0.693
$-i$	$0-1.570 i$	0
$-1.5 i$	$0.405-1.570 i$	0.405
$-2 i$	$0.693-1.570 i$	0.693

*The software used in this paper are Function Calculator (Gang, 2014), Draw Function Graphs – Plotter – Recheronline, Complex Number Calculator (Pierce, 2014) and MS-Excel.

The logarithm of i will be, $\log_{i,b}(i) = 0$, since, $i(b^0) = i$, and the logarithm of $-i$ will be, $\log_{-i,b}(-i) = 0$, since, $-i(b^0) = -i$.

Log natural of imaginary number coefficient, c . For $c = +i$ or $c = -i$, $\log_{c,e} c = 0$. That is,

$$\log_{+i,e}(+i) = 0 \text{ and } \log_{-i,e}(-i) = 0.$$

The inverse function of logarithm of natural positive imaginary logarithm will be $(+i)e^x$, $x > 0$. Similarly, the inverse function of logarithm of natural negative imaginary function will be $(-i)e^x$, $x > 0$, as given in figure 6.

A comparison of natural logarithms of imaginary numbers in polar and rectangular forms reveals that the real part of the two forms is equal (Table 1). The polar form logarithms have an imaginary part equal to $i(\pi/2)$ or $-i(\pi/2)$, for logarithms of $+ix$ or $-ix$ respectively.

2. Logarithmic operations on imaginary numbers

As per the present concept, logarithmic multiplication and division of imaginary and real numbers may be carried out.

Multiplication: For c_1 and c_2 same or different imaginary number coefficients ($c_1, c_2 = +i$ or $-i$)

$$\log_{c_1,b}(c_1x_1) + \log_{c_2,b}(c_2x_2) = \log_{c_1c_2,b}(c_1c_2x_1x_2), \quad x_1, x_2 > 0, b > 1. \quad (2.1)$$

For example if $c_1 = +i, c_2 = -i, b = 10, x_1 = 100, x_2 = 1000$
 $\log_{+i,10}(+i)100 + \log_{-i,10}(-i)1000 = \log_{-(i^2),b} -(i^2)100000$
 $= \log_{+1,b}(+1)100000$

Real and imaginary numbers can be presented in terms of the powers of imaginary numbers, such as,

$$(i^{-i} = -i), \quad (i^0 = +1), \quad (i^1 = +i), \quad (i^2 = -1), \quad (i^3 = -i), \quad (i^4 = +1).$$

Let ($c_1, c_2 = i^m, i^n$). Following general equation for the multiplication of real numbers and imaginary numbers is thus derived

$$\log_{i^m,b}(i^m x_1) + \log_{i^n,b}(i^n x_2) = \log_{i^{m+n},b}(i^{m+n} x_1 x_2), \quad x_1, x_2 > 0, b > 1. \quad (2.2)$$

Division: If c_1, c_2 are the same or different imaginary number coefficients, ($c_1, c_2 = +i$ or $-i$)

$$\log_{c_1,b}(c_1x_1) - \log_{c_2,b}(c_2x_2) = \log_{(c_1/c_2),b}(c_1x_1)/(c_2x_2), \quad x_1, x_2 > 0, b > 1$$

$$\log_{i^m,b}(i^m)x_1 - \log_{i^n,b}(i^n)x_2 = \log_{i^{m-n},b}(i^{m-n})x_1 / x_2 \quad (2.3)$$

Power: Log of power function, $c(x^n)$, $c = +i$ or $c = -i$ will be,

$$\log_{c,b} c(x^n) = n(\log_{c,b} cx), \quad x > 0, b > 1$$

For $c = +i$

$$\log_{+i,b} [(+i)(x^n)] = n[\log_{+i,b}(+i)(x)], \quad x > 0, b > 1$$

For $c = -i$

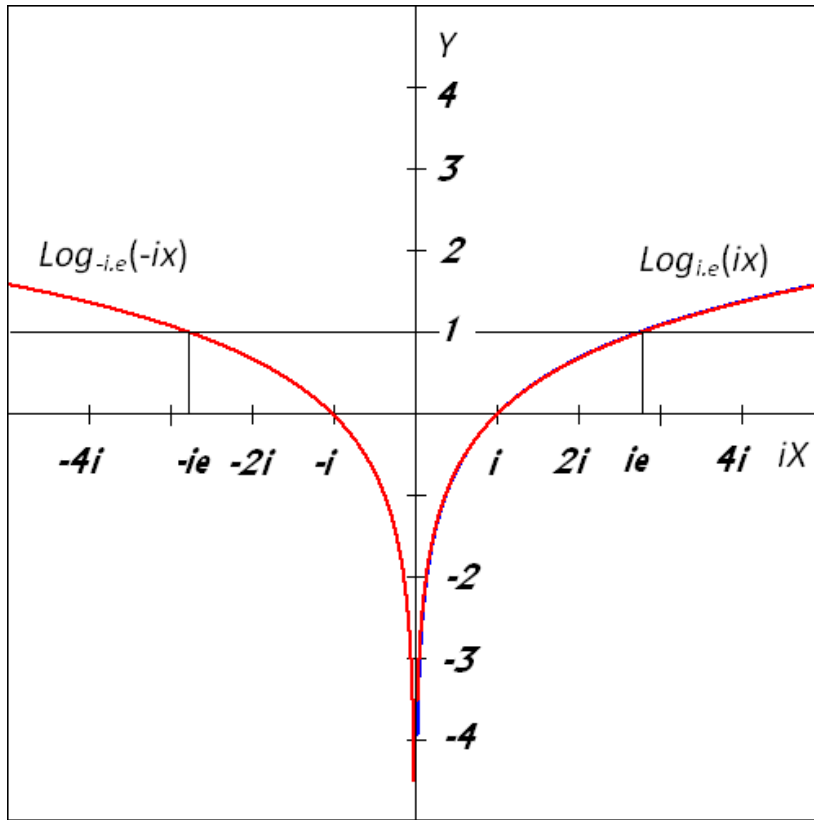


Fig. 5. Logarithmic curve for positive (right) and negative (left) imaginary numbers.

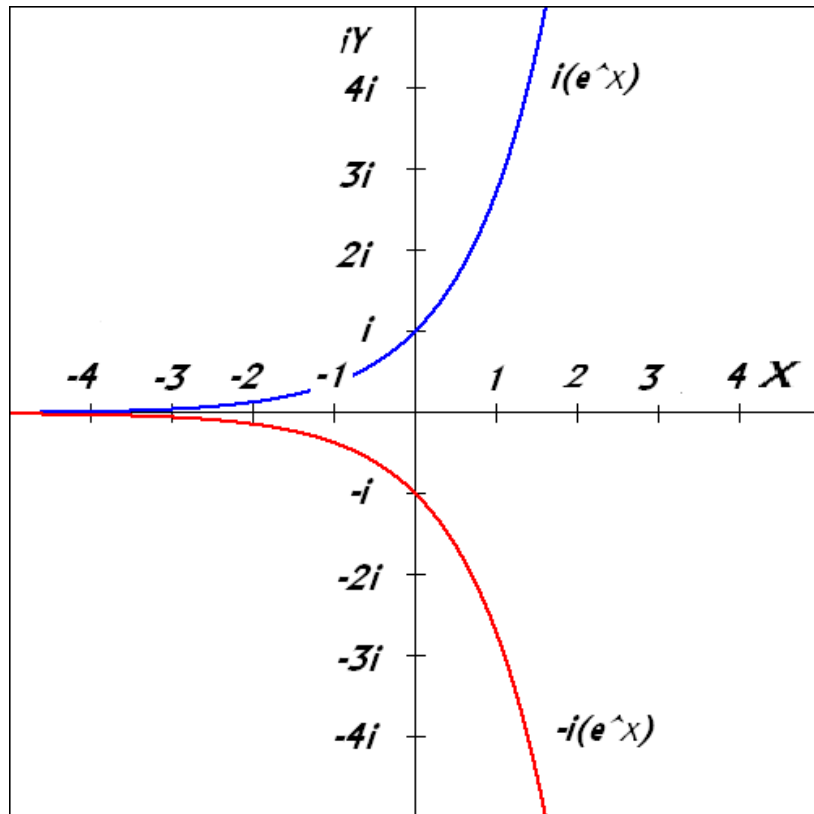


Fig. 6. Graph for positive (upper curve) and negative (lower curve) imaginary exponential functions.

$$\log_{-i,b}[(-i)(x^n)] = n[\log_{-i,b}(-i)x], x > 0, b > 1$$

Given below is an example involving the multiplication and division of real and imaginary numbers with the help of $\log_{c,e} cx$, where $b = e$; $c = +1, -1, +i$, or $-i$; and $x > 0$. The symbol * represents multiplication.

$$y = \frac{25 * (-10) * (+15i) * (-30i)}{50 * 20i},$$

$$\log_{c,e}(y) = \log_{1,e}(1 * 25) + \log_{-1,e}(-1 * 10) + \log_{i,e}(i * 15) + \log_{-i,e}(-i * 30) - \log_{1,e}(1 * 50) - \log_{i,e}(i * 20)$$

$$= 3.21887 + 2.30258 + 2.70805 + 3.4012 - 3.91202 - 2.99573 = 4.72295.$$

$$c = [1 * (-1) * i * (-i)] / [1 * i] = i.$$

$$y = i * \exp(4.72295) = 112.5i.$$

Therefore, the logarithms of imaginary numbers and real numbers in the rectangular form may provide an effective tool for calculations.

CONCLUSION

Logarithms of imaginary numbers were defined by Euler on a polar coordinate system with imaginary Y axis and real X axis. In the present paper, logarithms of imaginary numbers have been defined on a rectangular coordinate system consisting of both X and Y axes as imaginary axes. Hyperbolas formed in the IV and II quadrants may be used to define logarithms of positive and negative imaginary numbers respectively. Logarithms of imaginary numbers with (+i) or (-i) coefficient to the base are real numbers. This concept of logarithms will provide a useful technique to multiply and divide real and imaginary numbers using logarithms on a rectangular coordinate system, similar to the logarithms of real positive numbers.

ACKNOWLEDGEMENTS

Thanks are due to the University Grants Commission, and Department of Science and Technology, Government of India, New Delhi, for financial assistance.

REFERENCES

- Bal, D. 2014. Leibniz, Bernoulli and the logarithms of negative numbers. (<http://www.math.rverson.ca/~dbal/files/LeibBernLogs.pdf>)
- Cajori, F. 1913. History of the exponential and logarithmic concepts: The creation of a theory of logarithms of

complex numbers by Euler. The American Mathematical Monthly (Published by the Mathematical Association of America). 20:75-84.

(<http://www.jstor.org/stable/2973441>).

Clark, KM. and Montelle, C. 2011. Logarithms: The early history of a familiar function- John Napier introduces logarithms. Loci. pp. 1-6. (Published by the Mathematical Association of America). DOI:10.4169/loci003495. (<http://www.maa.org/publications/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-john-napier-introduces-logarithms>).

Euler, L. 1749. On the logarithms of negative and imaginary numbers. Acta. Acad. Berolinensis tomo VA. p. 139 Number 807 in the Enestrom index. Translation copy right © with 2005 Todd Doucet. (<http://eulerarchive.maa.org/docs/translations/E807en.pdf>)

Gang, XIAO. 2014. Function calculator. (http://wims.unice.fr/wims/en_tool~analysis~function.en.html).

Lefort, X. 2013. History of the logarithms. An example of the development of a concept in mathematics. pp. 1-9. (www.gobiernodecanarias.org).

Lexa, MA. 2013. Remembering John Napier and his logarithms. pp. 1-13. (www.see.ed.ac.uk/~mlexa/supportingdoc/mlexa_napier_revised.pdf).

Pierce, Rod. 2014. Complex Number Calculator. Math is Fun. (<http://www.mathsisfun.com/numbers/complex-number-calculator.html>).

Rechneronline. 2010. <http://rechneronline.de/function-graphs/>

Roegel, D. 2012. Napier's ideal construction of logarithms. LOCOMAT project, the LORIA Collection of Mathematical Tables. (<http://locomat.loria.fr/napier1619construction.pdf>).

Thukral, AK. and Parkash, O. 2014. A new approach for the logarithms of real negative numbers. Canadian Journal of Pure and Applied Sciences. 8(2):2955-2961.

Wikipedia. 2014^a. Logarithm. pp. 1-20. (<http://en.wikipedia.org/wiki/Logarithm>).

Wikipedia. 2014^b. Contributions of Leonhard Euler to mathematics. 1-6. (http://en.wikipedia.org/wiki/Contributions_of_Leonhard_Euler_to_mathematics).

Wikipedia. 2014^c. Complex number. pp. 1-25. ([En.wikipedia.org/wiki/Complex_number](http://en.wikipedia.org/wiki/Complex_number)).